

# The Loewner Framework for Model Reduction of Flow Equations

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# Motivation

- ▶ Full order model (FOM) (typically discretization of system of PDEs)

$$\mathbf{E} \frac{d}{dt} \mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{F}(\mathbf{y}(t)) + \mathbf{B}\mathbf{u}(t)$$

with state  $\mathbf{y}(t) \in \mathbb{R}^n$ ,  $n \gg 1$ , input  $\mathbf{u}(t) \in \mathbb{R}^m$ , and output

$$\mathbf{z}(t) = \mathbf{C}\mathbf{y}(t) + \mathbf{D}\mathbf{u}(t) \in \mathbb{R}^p.$$

- ▶ Projection based reduced order model (ROM):  
Matrices  $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ ,  $r \ll n$ ; ROM

$$\mathbf{W}^T \mathbf{E} \mathbf{V} \frac{d}{dt} \hat{\mathbf{y}}(t) = \mathbf{W}^T \mathbf{A} \mathbf{V} \hat{\mathbf{y}}(t) + \mathbf{W}^T \mathbf{F}(\mathbf{V} \hat{\mathbf{y}}(t)) + \mathbf{W}^T \mathbf{B} \mathbf{u}(t)$$

with state  $\hat{\mathbf{y}}(t) \in \mathbb{R}^r$ ,  $r \ll n$ , input  $\mathbf{u}(t) \in \mathbb{R}^m$ , and output

$$\hat{\mathbf{z}}(t) = \mathbf{C} \mathbf{V} \hat{\mathbf{y}}(t) + \mathbf{D} \mathbf{u}(t) \in \mathbb{R}^p.$$

- ▶ Goal: Input-to-output map  $\mathbf{u} \mapsto \hat{\mathbf{z}}$  of ROM approximates input-to-output map  $\mathbf{u} \mapsto \mathbf{z}$  of FOM.
- ▶ Notation: States  $\mathbf{y}$ , input  $\mathbf{u}$ , output  $\mathbf{z}$ .

Many ROM methods, incl.

- ▶ Proper Orthogonal Decomposition (POD) (e.g., books and survey articles [Gubisch and Volkwein, 2017], [Hesthaven et al., 2015], [Quarteroni et al., 2016]).
- ▶ Reduced Basis (RB) methods (e.g., books and survey articles [Haasdonk, 2017], [Hesthaven et al., 2015], [Quarteroni et al., 2016], [Rozza et al., 2008]).
- ▶ Balanced Truncation and Balanced POD (e.g., book [Antoulas, 2005] and survey articles [Benner and Breiten, 2017], [Rowley and Dawson, 2017]).
- ▶ Rational Interpolation (e.g., book [Antoulas et al., 2020a] and survey article [Beattie and Gugercin, 2017]).

All require access to  $\mathbf{E}, \mathbf{A}, \dots$  to generate projection matrices  
 $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ , then generate ROM  $\hat{\mathbf{E}} = \mathbf{W}^T \mathbf{E} \mathbf{V}$ ,  $\hat{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{V}, \dots$

### This talk: Loewner framework

- ▶ Constructs ROM  $\hat{\mathbf{E}}, \hat{\mathbf{A}}, \dots$  directly from data.
- ▶ For LTI systems [Antoulas et al., 2020a], [Antoulas et al., 2017]; recent work on descriptor systems, incl. Oseen equations (e.g., [Antoulas et al., 2020b], [Gosea et al., 2020]).
- ▶ Extension to quadratic-bilinear systems [Gosea and Antoulas, 2018], Burger's equation [Antoulas et al., 2018].

# Loewner for Linear Time-Invariant (LTI) Systems

- ▶ Consider linear time-invariant system

$$\mathbf{E} \frac{d}{dt} \mathbf{y}(t) = \mathbf{A} \mathbf{y}(t) + \mathbf{b} \mathbf{u}(t)$$

with state  $\mathbf{y}(t) \in \mathbb{R}^n$ ,  $n \gg 1$ , input  $\mathbf{u}(t) \in \mathbb{R}$ , and output

$$\mathbf{z}(t) = \mathbf{c}^T \mathbf{y}(t) + d \mathbf{u}(t) \in \mathbb{R}.$$

Example: Oseen equations.

- ▶ To simplify presentation consider case of single input  $\mathbf{b} \in \mathbb{R}^n$ ,  $\mathbf{u}(t) \in \mathbb{R}$ , and single output  $\mathbf{c} \in \mathbb{R}^n$ ,  $\mathbf{z}(t) \in \mathbb{R}$  (SISO).

Multiple inputs, multiple outputs (MIMO) can be handled via so-called left and right tangential directions, but is a bit more technical and requires more notation.

# Frequency Domain

- ▶ Frequency domain  $s \in i\mathbb{R}$

$$s \mathbf{E} \mathbf{y}(s) = \mathbf{A} \mathbf{y}(s) + \mathbf{b} \mathbf{u}(s), \\ \mathbf{z}(s) = \mathbf{c}^T \mathbf{y}(s).$$

From now on we work in frequency domain;  
use same notation  $\mathbf{y}$ ,  $\mathbf{u}$ ,  $\mathbf{z}$  for variables in frequency domain.

- ▶ Transfer function

$$\mathbf{H}(s) = \mathbf{c}^T (s\mathbf{E} - \mathbf{A})^{-1} \mathbf{b}.$$

- ▶ Seek ROM  $\widehat{\mathbf{E}} = \mathbf{W}^T \mathbf{E} \mathbf{V}$ ,  $\widehat{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{V}$ ,  $\widehat{\mathbf{b}} = \mathbf{W}^T \mathbf{b}$ ,  $\widehat{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$ ,

$$s \widehat{\mathbf{E}} \widehat{\mathbf{y}}(s) = \widehat{\mathbf{A}} \widehat{\mathbf{y}}(s) + \widehat{\mathbf{b}} \mathbf{u}(s), \\ \widehat{\mathbf{z}}(s) = \widehat{\mathbf{c}}^T \widehat{\mathbf{y}}(s) + d \mathbf{u}(s),$$

so that transfer function

$$\widehat{\mathbf{H}}(s) = \widehat{\mathbf{c}}^T (s \widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1} \widehat{\mathbf{b}} \approx \mathbf{H}(s) = \mathbf{c}^T (s\mathbf{E} - \mathbf{A})^{-1} \mathbf{b}.$$

# Interpolation

- ▶ Given distinct frequencies  $\mu_1, \dots, \mu_r, \lambda_1, \dots, \lambda_r \in \mathbb{C}$ , want ROM s.t.

$$\widehat{\mathbf{c}}^T(\mu_j \widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1} \widehat{\mathbf{b}} = \widehat{\mathbf{H}}(\mu_j) = \mathbf{H}(\mu_j) = \mathbf{c}^T(\mu_j \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \quad j = 1, \dots, r,$$
$$\widehat{\mathbf{c}}^T(\lambda_j \widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1} \widehat{\mathbf{b}} = \widehat{\mathbf{H}}(\lambda_j) = \mathbf{H}(\lambda_j) = \mathbf{c}^T(\lambda_j \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \quad j = 1, \dots, r.$$

- ▶ Theorem: Interpolation guaranteed if ROM projection matrices  $\mathbf{V}, \mathbf{W}$  satisfy

$$(\lambda_j \mathbf{E} - \mathbf{A})^{-1} \mathbf{b} \in R(\mathbf{V}), \quad j = 1, \dots, r,$$

$$(\mathbf{c}^T(\mu_j \mathbf{E} - \mathbf{A})^{-1})^T \in R(\mathbf{W}), \quad j = 1, \dots, r.$$

- ▶ Can choose (assume these have full rank)

$$\mathbf{V} = \left[ (\lambda_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \dots, (\lambda_r \mathbf{E} - \mathbf{A})^{-1} \mathbf{b} \right] \in \mathbb{C}^{n \times r},$$

$$\mathbf{W}^* = \left[ \begin{array}{c} \mathbf{c}^T(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{c}^T(\mu_r \mathbf{E} - \mathbf{A})^{-1} \end{array} \right] \in \mathbb{C}^{r \times n}.$$

## Elementary identities

Recall  $\mathbf{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$ .

$$\mathbf{c}^T(\mu\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

$$= \frac{1}{\mu - \lambda} \left( \mathbf{c}^T(\mu\mathbf{E} - \mathbf{A})^{-1}(\mu - \lambda)\mathbf{E}(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} \right)$$

$$= \frac{1}{\mu - \lambda} \left( \mathbf{c}^T(\mu\mathbf{E} - \mathbf{A})^{-1} \left( (\mu\mathbf{E} - \mathbf{A}) - (\lambda\mathbf{E} - \mathbf{A}) \right) (\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} \right)$$

$$= \frac{1}{\mu - \lambda} \left( \mathbf{H}(\lambda) - \mathbf{H}(\mu) \right).$$

$$\mathbf{c}^T(\mu\mathbf{E} - \mathbf{A})^{-1}\mathbf{A}(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

$$= -\mathbf{c}^T(\mu\mathbf{E} - \mathbf{A})^{-1}(\mu\mathbf{E} - \mathbf{A})(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} + \mu\mathbf{c}^T(\mu\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

$$= -\mathbf{H}(\lambda) + \mu\mathbf{c}^T(\mu\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}(\lambda\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

$$= -\mathbf{H}(\lambda) + \mu \frac{1}{\mu - \lambda} \left( \mathbf{H}(\lambda) - \mathbf{H}(\mu) \right)$$

$$= \frac{1}{\mu - \lambda} \left( \lambda\mathbf{H}(\lambda) - \mu\mathbf{H}(\mu) \right).$$

If  $\mathbf{V} = \left[ (\lambda_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \dots, (\lambda_r \mathbf{E} - \mathbf{A})^{-1} \mathbf{b} \right] \in \mathbb{C}^{n \times r}$ ,

$$\mathbf{W}^* = \begin{bmatrix} \mathbf{c}^T(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{c}^T(\mu_r \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix} \in \mathbb{C}^{r \times n},$$

then (recall  $\mathbf{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1} \mathbf{b}$ )

$$\widehat{\mathbf{E}} = \mathbf{W}^* \mathbf{E} \mathbf{V} = - \begin{bmatrix} \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1)}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_r)}{\mu_1 - \lambda_r} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_r) - \mathbf{H}(\lambda_1)}{\mu_r - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_r) - \mathbf{H}(\lambda_r)}{\mu_r - \lambda_r} \end{bmatrix} \stackrel{\text{def}}{=} -\mathbb{L},$$

$$\widehat{\mathbf{A}} = \mathbf{W}^* \mathbf{A} \mathbf{V} = - \begin{bmatrix} \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_1 - \lambda_1} & \dots & \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_r) \lambda_r}{\mu_1 - \lambda_r} \\ \vdots & \ddots & \vdots \\ \frac{\mu_r \mathbf{H}(\mu_r) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_r - \lambda_1} & \dots & \frac{\mu_r \mathbf{H}(\mu_r) - \mathbf{H}(\lambda_r) \lambda_r}{\mu_r - \lambda_r} \end{bmatrix} \stackrel{\text{def}}{=} -\mathbb{L}_s,$$

$$\widehat{\mathbf{b}} = \mathbf{W}^* \mathbf{b} = [\mathbf{H}(\mu_1), \dots, \mathbf{H}(\mu_r)]^T, \quad \widehat{\mathbf{c}} = \mathbf{V}^* \mathbf{c} = [\mathbf{H}(\lambda_1), \dots, \mathbf{H}(\lambda_r)]^T.$$

If  $\mathbf{V} = \left[ (\lambda_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \dots, (\lambda_r \mathbf{E} - \mathbf{A})^{-1} \mathbf{b} \right] \in \mathbb{C}^{n \times r}$ ,

$$\mathbf{W}^* = \begin{bmatrix} \mathbf{c}^T(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{c}^T(\mu_r \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix} \in \mathbb{C}^{r \times n},$$

then (recall  $\mathbf{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$ )

$$\widehat{\mathbf{E}} = \mathbf{W}^* \mathbf{E} \mathbf{V} = - \begin{bmatrix} \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1)}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_r)}{\mu_1 - \lambda_r} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_r) - \mathbf{H}(\lambda_1)}{\mu_r - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_r) - \mathbf{H}(\lambda_r)}{\mu_r - \lambda_r} \end{bmatrix} \stackrel{\text{def}}{=} -\mathbb{L},$$

$$\widehat{\mathbf{A}} = \mathbf{W}^* \mathbf{A} \mathbf{V} = - \begin{bmatrix} \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_1 - \lambda_1} & \dots & \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_r) \lambda_r}{\mu_1 - \lambda_r} \\ \vdots & \ddots & \vdots \\ \frac{\mu_r \mathbf{H}(\mu_r) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_r - \lambda_1} & \dots & \frac{\mu_r \mathbf{H}(\mu_r) - \mathbf{H}(\lambda_r) \lambda_r}{\mu_r - \lambda_r} \end{bmatrix} \stackrel{\text{def}}{=} -\mathbb{L}_s,$$

$$\widehat{\mathbf{b}} = \mathbf{W}^* \mathbf{b} = [\mathbf{H}(\mu_1), \dots, \mathbf{H}(\mu_r)]^T, \quad \widehat{\mathbf{c}} = \mathbf{V}^* \mathbf{c} = [\mathbf{H}(\lambda_1), \dots, \mathbf{H}(\lambda_r)]^T.$$

**Only need data  $(\mu_j, \mathbf{H}(\mu_j)), (\lambda_j, \mathbf{H}(\lambda_j))$ ,  $j = 1, \dots, r$ ,  
to construct ROM  $\widehat{\mathbf{E}}, \widehat{\mathbf{A}}, \widehat{\mathbf{b}}, \widehat{\mathbf{c}}$ !**

# Loewner Framework

- ▶ Given distinct frequencies  $\mu_1, \dots, \mu_k, \lambda_1, \dots, \lambda_k \in \mathbb{C}$  and transfer function measurements  $\mathbf{H}(\mu_1), \dots, \mathbf{H}(\mu_k), \mathbf{H}(\lambda_1), \dots, \mathbf{H}(\lambda_k) \in \mathbb{C}$ ,
- ▶ use Loewner matrix

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1)}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_k)}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_k) - \mathbf{H}(\lambda_1)}{\mu_k - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_k) - \mathbf{H}(\lambda_k)}{\mu_k - \lambda_k} \end{bmatrix} \in \mathbb{C}^{k \times k}$$

and shifted Loewner matrix

$$\mathbb{L}_s = \begin{bmatrix} \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_1 - \lambda_1} & \dots & \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_k) \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_k \mathbf{H}(\mu_k) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_k - \lambda_1} & \dots & \frac{\mu_k \mathbf{H}(\mu_k) - \mathbf{H}(\lambda_k) \lambda_k}{\mu_k - \lambda_k} \end{bmatrix} \in \mathbb{C}^{k \times k}$$

- ▶ to directly compute ROM  $\widehat{\mathbf{E}}, \widehat{\mathbf{A}} \in \mathbb{R}^{r \times r}, \widehat{\mathbf{b}}, \widehat{\mathbf{c}} \in \mathbb{R}^r$ , such that
- $$\widehat{\mathbf{c}}^T (\mu_j \widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1} \widehat{\mathbf{b}} = \widehat{\mathbf{H}}(\mu_j) \approx \mathbf{H}(\mu_j) = \mathbf{c}^T (\mu_j \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \quad j = 1, \dots, k,$$
- $$\widehat{\mathbf{c}}^T (\lambda_j \widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1} \widehat{\mathbf{b}} = \widehat{\mathbf{H}}(\lambda_j) \approx \mathbf{H}(\lambda_j) = \mathbf{c}^T (\lambda_j \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \quad j = 1, \dots, k.$$

# Loewner Framework

- ▶ Given distinct frequencies  $\mu_1, \dots, \mu_k, \lambda_1, \dots, \lambda_k \in \mathbb{C}$  and transfer function measurements  $\mathbf{H}(\mu_1), \dots, \mathbf{H}(\mu_k), \mathbf{H}(\lambda_1), \dots, \mathbf{H}(\lambda_k) \in \mathbb{C}$ ,
- ▶ use Loewner matrix

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1)}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_1) - \mathbf{H}(\lambda_k)}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_k) - \mathbf{H}(\lambda_1)}{\mu_k - \lambda_1} & \dots & \frac{\mathbf{H}(\mu_k) - \mathbf{H}(\lambda_k)}{\mu_k - \lambda_k} \end{bmatrix} \in \mathbb{C}^{k \times k}$$

and shifted Loewner matrix

$$\mathbb{L}_s = \begin{bmatrix} \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_1 - \lambda_1} & \dots & \frac{\mu_1 \mathbf{H}(\mu_1) - \mathbf{H}(\lambda_k) \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_k \mathbf{H}(\mu_k) - \mathbf{H}(\lambda_1) \lambda_1}{\mu_k - \lambda_1} & \dots & \frac{\mu_k \mathbf{H}(\mu_k) - \mathbf{H}(\lambda_k) \lambda_k}{\mu_k - \lambda_k} \end{bmatrix} \in \mathbb{C}^{k \times k}$$

- ▶ to directly compute ROM  $\widehat{\mathbf{E}}, \widehat{\mathbf{A}} \in \mathbb{R}^{r \times r}$ ,  $\widehat{\mathbf{b}}, \widehat{\mathbf{c}} \in \mathbb{R}^r$ , such that
  - $\widehat{\mathbf{c}}^T (\mu_j \widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1} \widehat{\mathbf{b}} = \widehat{\mathbf{H}}(\mu_j) \approx \mathbf{H}(\mu_j) = \mathbf{c}^T (\mu_j \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}$ ,  $j = 1, \dots, k$ ,
  - $\widehat{\mathbf{c}}^T (\lambda_j \widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1} \widehat{\mathbf{b}} = \widehat{\mathbf{H}}(\lambda_j) \approx \mathbf{H}(\lambda_j) = \mathbf{c}^T (\lambda_j \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}$ ,  $j = 1, \dots, k$ .
- ▶ Typically more data  $k$  than ROM size  $r$ .

# Loewner Framework: Redundant Data

- ▶ Given distinct frequencies  $\mu_1, \dots, \mu_k, \lambda_1, \dots, \lambda_k \in \mathbb{C}$  and transfer function measurements  $\mathbf{H}(\mu_1), \dots, \mathbf{H}(\mu_k), \mathbf{H}(\lambda_1), \dots, \mathbf{H}(\lambda_k) \in \mathbb{C}$ .
- ▶ Compute Loewner  $\mathbb{L} \in \mathbb{C}^{k \times k}$  and shifted Loewner  $\mathbb{L}_s \in \mathbb{C}^{k \times k}$ .
- ▶ Compute (short) SVDs

$$[\mathbb{L} \quad \mathbb{L}_s] = \mathbf{Y}_1 \Sigma_1 \mathbf{X}_1^*, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} = \mathbf{Y}_2 \Sigma_2 \mathbf{X}_2^*,$$

where  $\Sigma_1 \in \mathbb{R}^{k \times 2k}$ ,  $\Sigma_2 \in \mathbb{R}^{2k \times k}$ ,  $\mathbf{Y}_1, \mathbf{X}_2 \in \mathbb{C}^{k \times k}$ .

- ▶  $\mathbf{Y}, \mathbf{X} \in \mathbb{C}^{k \times r}$  obtained by selecting first  $r$  columns of  $\mathbf{Y}_1$  and  $\mathbf{X}_2$ . ( $r$  determined based on singular values.)
- ▶ ROM:

$$\widehat{\mathbf{E}} = -\mathbf{Y}^* \mathbb{L} \mathbf{X}, \quad \widehat{\mathbf{A}} = -\mathbf{Y}^* \mathbb{L}_s \mathbf{X},$$
$$\widehat{\mathbf{b}} = \mathbf{Y}^* (\mathbf{H}(\mu_1), \dots, \mathbf{H}(\mu_k))^T, \quad \widehat{\mathbf{c}}^T = (\mathbf{H}(\lambda_1), \dots, \mathbf{H}(\lambda_k)) \mathbf{X}.$$

## Loewner Framework: Complex $\rightarrow$ Real

- ▶ Data  $\mu_j, \lambda_j, \dots$  complex  $\Rightarrow$  previous ROM  $\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$  complex.
- ▶ If data also includes conjugate complex data,

$$\{\mu_j\}_{j=1}^k = \{\bar{\mu}_j\}_{j=1}^k, \quad \{\lambda_j\}_{j=1}^k = \{\bar{\lambda}_j\}_{j=1}^k,$$

then can transform complex ROM into real ROM with same transfer function.

# Loewner Framework: Transfer Function at Infinity

- So far considered FOMs

$$s \mathbf{E} \mathbf{y}(s) = \mathbf{A} \mathbf{y}(s) + \mathbf{b} \mathbf{u}(s), \\ \mathbf{z}(s) = \mathbf{c}^T \mathbf{y}(s)$$

with transfer function  $\mathbf{H}(s) = \mathbf{H}_{\text{spr}}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$

- However, in general transfer function composed of strictly proper part and polynomial part

$$\mathbf{H}(s) = \mathbf{H}_{\text{spr}}(s) + \mathbf{H}_{\text{poly}}(s),$$

where  $\mathbf{H}_{\text{spr}}(s) \rightarrow 0$  as  $|s| \rightarrow \infty$ , and  $\mathbf{H}_{\text{poly}}(s) = d_1 + d_2 s + \dots$ .

In particular discretized Oseen has  $\mathbf{H}_{\text{poly}}(s) \neq 0$ .

- In practice,  $\hat{\mathbf{H}}(s) \approx \mathbf{H}(s)$  in range of frequency data, not outside.  
Loewner ROM generates transfer function

$$\hat{\mathbf{H}}(s) = \hat{\mathbf{H}}_{\text{spr}}(s) \quad (\hat{\mathbf{H}}_{\text{poly}}(s) = 0).$$

- Can compute  $\mathbf{H}_{\text{poly}}(s)$  or estimate (Loewner with high freq. data), and apply Lowener with measurements

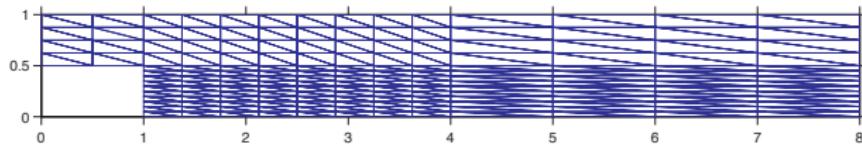
$$\mathbf{H}_{\text{spr}}(s) = \mathbf{H}(s) - \mathbf{H}_{\text{poly}}(s),$$

to generate a ROM with transfer function

# Oseen Equations

$$\begin{aligned}\frac{\partial}{\partial t}v(x,t) + (a(x) \cdot \nabla)v(x,t) - \nu\Delta v(x,t) + \nabla p(x,t) &= 0 && \text{in } \Omega \times (0,T), \\ \nabla \cdot v(x,t) &= 0 && \text{in } \Omega \times (0,T), \\ (-p(x,t)I + \nu\nabla v(x,t))n(x) &= 0 && \text{on } \Gamma_n \times (0,T), \\ v(x,t) &= 0 && \text{on } \Gamma_d \times (0,T), \\ v(x,t) &= g(x,t) && \text{on } \Gamma_g \times (0,T), \\ v(x,0) &= 0 && \text{in } \Omega,\end{aligned}$$

with  $\nu = 1/50$  and domain



and 6 inputs (suction blowing at inflow, backward facing step).

Output:(weak form approximation of)

$$z(t) = \int_{\Gamma_{\text{obs}}} (-p(x,t)I + \nu\nabla v(x,t))n(x)ds \quad (\text{2 outputs}).$$

# Taylor-Hood FEM Semidiscretization of Oseen Equations

$$\begin{aligned}\mathbf{E}_{11} \frac{d}{dt} \mathbf{v}(t) &= \mathbf{A}_{11} \mathbf{v}(t) + \mathbf{A}_{12} \mathbf{p}(t) + \mathbf{B}_{10} \mathbf{g}(t) + \mathbf{B}_{11} \frac{d}{dt} \mathbf{g}(t), \quad t \in (0, T), \\ \mathbf{0} &= \mathbf{A}_{12}^T \mathbf{v}(t) + \mathbf{B}_{20} \mathbf{g}(t), \quad t \in (0, T), \\ \mathbf{v}(0) &= \mathbf{0}, \\ \mathbf{z}(t) &= \mathbf{C}_1 \mathbf{v}(t) + \mathbf{C}_2 \mathbf{p}(t) + \mathbf{D}_0 \mathbf{g}(t) + \mathbf{D}_1 \frac{d}{dt} \mathbf{g}(t), \quad t \in (0, T).\end{aligned}$$

$\frac{d}{dt} \mathbf{g}(t)$  terms arise, e.g., when Dirichlet control inputs are applied.

Transfer function

$$\mathbf{H}(s) = \mathbf{H}_{\text{spr}}(s) + \mathbf{P}_0 + s \mathbf{P}_1,$$

with

$$\begin{aligned}\mathbf{P}_0 &= \mathbf{D}_0 - \mathbf{C}_1 \mathbf{E}_{11}^{-1} \mathbf{A}_{12} (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} \mathbf{B}_{2,0} - \mathbf{C}_2 (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} \mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{B}_3, \\ \mathbf{P}_1 &= \mathbf{D}_1 - \mathbf{C}_2 (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{B}_{1,1} + \mathbf{B}_{2,0}),\end{aligned}$$

where  $\mathbf{B}_3 := \mathbf{B}_{1,0} - \mathbf{A}_{11} \mathbf{E}_{11}^{-1} \mathbf{A}_{12} (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} \mathbf{B}_{2,0}$ .

► Transfer function

$$\mathbf{H}(s) = \mathbf{H}_{\text{spr}}(s) + \mathbf{P}_0 + s \mathbf{P}_1,$$

has polynomials part induced by algebraic constraint

$$\begin{aligned}\mathbf{P}_0 &= \mathbf{D}_0 - \mathbf{C}_1 \mathbf{E}_{11}^{-1} \mathbf{A}_{12} (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} \mathbf{B}_{2,0} - \mathbf{C}_2 (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} \mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{B}_3, \\ \mathbf{P}_1 &= \mathbf{D}_1 - \mathbf{C}_2 (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{B}_{1,1} + \mathbf{B}_{2,0}),\end{aligned}$$

$$\text{where } \mathbf{B}_3 := \mathbf{B}_{1,0} - \mathbf{A}_{11} \mathbf{E}_{11}^{-1} \mathbf{A}_{12} (\mathbf{A}_{12}^T \mathbf{E}_{11}^{-1} \mathbf{A}_{12})^{-1} \mathbf{B}_{2,0}.$$

- Tedious to compute.
- Can estimate  $\mathbf{P}_0$  and  $\mathbf{P}_1$  from high frequency data.
- Instead of applying Loewner to  $\mathbf{H}(s) = \mathbf{H}_{\text{spr}}(s) + \mathbf{P}_0 + s \mathbf{P}_1$  apply Loewner to (estimated) strictly proper part  $\mathbf{H}(s) - \widehat{\mathbf{P}}_0 + s \widehat{\mathbf{P}}_1$ .

# Estimation of $\mathbf{P}_0$ and $\mathbf{P}_1$ in $\mathbf{H}(s) = \mathbf{H}_{\text{spr}}(s) + \mathbf{P}_0 + s \mathbf{P}_1$

- From one frequency  $\eta \gg 1$

$$\mathbf{H}(\imath \eta) = \underbrace{\mathbf{H}_{\text{spr}}(\imath \eta)}_{\rightarrow 0 (\eta \rightarrow \infty)} + \mathbf{P}_0 + \imath \eta \mathbf{P}_1 \approx \mathbf{P}_0 + \imath \eta \mathbf{P}_1.$$

Estimates  $\widehat{\mathbf{P}}_0 = \Re(\mathbf{H}(\imath \eta))$  and  $\widehat{\mathbf{P}}_1 = \eta^{-1} \Im(\mathbf{H}(\imath \eta))$ .

- From two frequencies  $\theta > \eta \gg 1$ .

$$\begin{aligned} \mathbf{H}(\imath \theta) - \mathbf{H}(\imath \eta) &= \left( \mathbf{H}_{\text{spr}}(\imath \theta) + \mathbf{P}_0 + \imath \theta \mathbf{P}_1 \right) - \left( \mathbf{H}_{\text{spr}}(\imath \eta) + \mathbf{P}_0 + \imath \eta \mathbf{P}_1 \right) \\ &= \mathbf{H}_{\text{spr}}(\imath \theta) - \mathbf{H}_{\text{spr}}(\imath \eta) + (\imath \theta - \imath \eta) \mathbf{P}_1 \approx (\imath \theta - \imath \eta) \mathbf{P}_1. \end{aligned}$$

Estimate

$$\widehat{\mathbf{P}}_1 = \Re\left(\frac{\mathbf{H}(\imath \theta) - \mathbf{H}(\imath \eta)}{\imath \theta - \imath \eta}\right) \quad \dots \text{ and } \widehat{\mathbf{P}}_0 = \Re\left(\frac{\imath \theta \mathbf{H}(\imath \theta) - \imath \eta \mathbf{H}(\imath \eta)}{\imath \theta - \imath \eta}\right).$$

- Arbitrary number of (high) frequencies.  
Form Loewner  $\mathbb{L}^{\text{hi}}$  and shifted Loewner  $\mathbb{L}_s^{\text{hi}}$ . Estimate

$$\widehat{\mathbf{P}}_0 = \Re\left((\mathbf{L}^{\text{hi}})^\dagger \mathbb{L}_s^{\text{hi}} (\mathbf{R}^{\text{hi}})^\dagger\right), \quad \widehat{\mathbf{P}}_1 = \Re\left((\mathbf{L}^{\text{hi}})^\dagger \mathbb{L}^{\text{hi}} (\mathbf{R}^{\text{hi}})^\dagger\right),$$

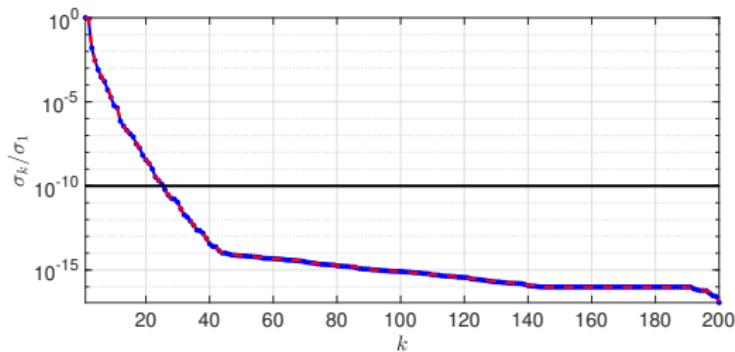
( $\mathbf{L}^{\text{hi}}$ ,  $\mathbf{R}^{\text{hi}}$  matrices of directions.)

## Full Order Model

# velocities  $n_v = 12,504$   
# pressures  $n_p = 1,669$   
# inputs  $m = 6$   
# outputs  $p = 2$

## Loewner ROM

Data: 20 logarithmically spaced points between  $10^3$  and  $10^5$  and their complex conjugates (will be able to construct real ROMs) to estimate  $\mathbf{P}_0$  and  $\mathbf{P}_1$ .  
100 logarithmically spaced points between  $10^{-2}$  and  $10^1$  and their complex conjugates to compute ROM

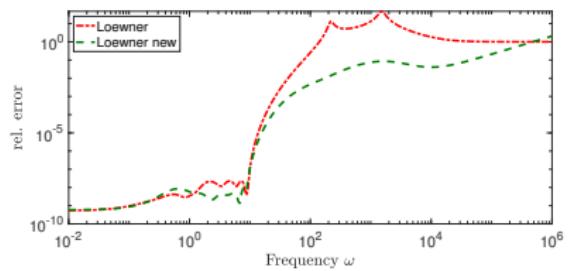
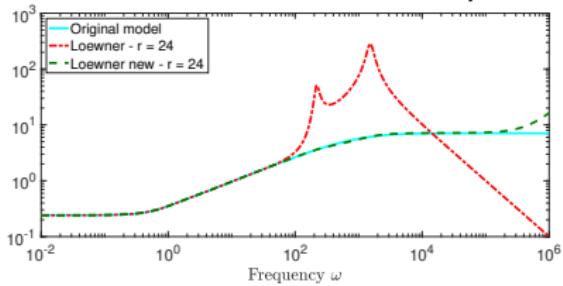


Normalized singular values of Loewner matrices

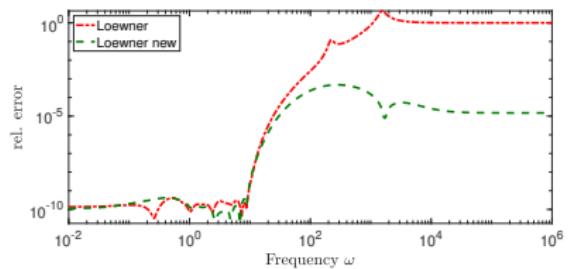
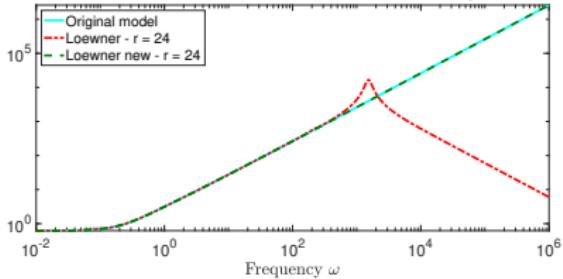
$$[\mathbb{L} \quad \mathbb{L}_s] \text{ and } \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix}.$$

ROM size  $r = 24$  is smallest  $r$  with  $\sigma_r/\sigma_1 > 10^{-10}$ .

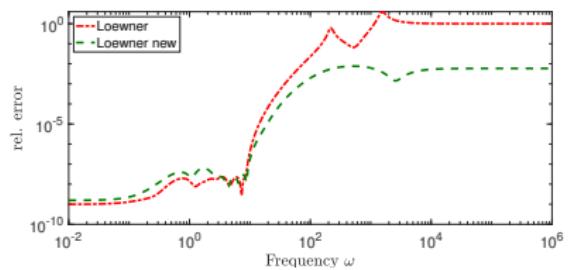
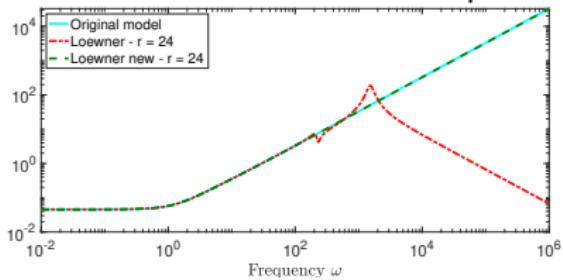
# Input 1 - Output 1



# Input 3 - Output 2



# Input 5 - Output 2



# Loewner for Quadratic-Bilinear Systems

Consider quadratic-bilinear SISO system

$$\begin{aligned}\mathbf{E} \frac{d}{dt} \mathbf{y}(t) &= \mathbf{A} \mathbf{y}(t) + \mathbf{G}(\mathbf{y}(t), \mathbf{y}(t)) + \mathbf{N} \mathbf{y}(t) \mathbf{u}(t) + \mathbf{b} \mathbf{u}(t), \quad t \in (0, T), \\ \mathbf{z}(t) &= \mathbf{c}^T \mathbf{y}(t) + d \mathbf{u}(t), \quad t \in (0, T), \\ \mathbf{y}(0) &= \mathbf{0},\end{aligned}$$

$\mathbf{E}, \mathbf{A}, \mathbf{N} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ ,  $d \in \mathbb{R}$ , and  $\mathbf{G} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is bilinear.

System called **quadratic-bilinear** because  $\mathbf{y} \mapsto \mathbf{G}(\mathbf{y}, \mathbf{y})$  is quadratic and  $(\mathbf{y}, \mathbf{u}) \mapsto \mathbf{N} \mathbf{y} \mathbf{u}$  is bilinear.

Again present SISO case, but can extend to MIMO.

Idea for ROM construction

- ▶ Expand quadratic-bilinear system into series of linear systems (e.g., [Rugh, 1981]).
- ▶ Identify transfer functions from expanded linear systems.
- ▶ Extend ROM methods for LTI system to generate ROM that approximates these transfer functions.
- ▶ Other approaches, e.g., [Benner and Breiten, 2015], [Ahmad et al., 2017].
- ▶ Loewner [Antoulas et al., 2018].

# Expansion of Quadratic-Bilinear Systems

- ▶ Expand around  $\mathbf{u}_0(t) = \mathbf{0}$  (corresponding state  $\mathbf{y}_0(t) = \mathbf{0}$ ).
- ▶ Assume control  $\mathbf{u}_0(t) + a\mathbf{u}(t)$  for scalar  $a > 0$ .

Corresponding state is of form

$$\mathbf{y}(t) = \sum_{\ell=1}^{\infty} a^{\ell} \mathbf{y}_{\ell}(t)$$

and output is

$$\mathbf{z}(t) = \mathbf{c}^T \mathbf{y}(t) + d(\mathbf{u}_0(t) + a\mathbf{u}(t)) = \sum_{\ell=1}^{\infty} a^{\ell} \mathbf{c}^T \mathbf{y}_{\ell}(t) + ad\mathbf{u}(t).$$

- ▶ States  $\mathbf{y}_{\ell}$  in expansion satisfy

$$\mathbf{E} \frac{d}{dt} \mathbf{y}_1(t) = \mathbf{A} \mathbf{y}_1(t) + \mathbf{b} \mathbf{u}(t), \quad \ell = 1,$$

$$\mathbf{E} \frac{d}{dt} \mathbf{y}_2(t) = \mathbf{A} \mathbf{y}_2(t) + \mathbf{G}(\mathbf{y}_1(t), \mathbf{y}_1(t)) + \mathbf{N} \mathbf{y}_1(t) \mathbf{u}(t), \quad \ell = 2,$$

$$\begin{aligned} \mathbf{E} \frac{d}{dt} \mathbf{y}_3(t) = \mathbf{A} \mathbf{y}_3(t) + \mathbf{G}(\mathbf{y}_1(t), \mathbf{y}_2(t)) + \mathbf{G}(\mathbf{y}_2(t), \mathbf{y}_1(t)) \\ + \mathbf{N} \mathbf{y}_2(t) \mathbf{u}(t), \quad \ell = 3, \end{aligned}$$

⋮

Given  $\mathbf{y}_1, \dots, \mathbf{y}_{\ell-1}$ ,  $\ell$ -th equation is linear in  $\mathbf{y}_{\ell}$ .

- If  $\mathbf{E}$  is invertible (assume  $\mathbf{E} = I$  to shorten notation) solutions are

$$\mathbf{y}_1(t) = \int_0^t e^{\mathbf{A}\tau_1} \mathbf{b} \mathbf{u}(t - \tau_1) d\tau_1,$$

$$\mathbf{y}_2(t) = \int_0^t e^{\mathbf{A}\tau_2} [\mathbf{G}(\mathbf{y}_1(t - \tau_2), \mathbf{y}_1(t - \tau_2)) + \mathbf{N}\mathbf{y}_1(t - \tau_2)\mathbf{u}(t - \tau_2)] d\tau_2,$$

⋮

- Output - here truncated at  $\ell = 2$ :

$$\begin{aligned} \mathbf{z}(t) &= \sum_{\ell=1}^{\infty} a^{\ell} \mathbf{c}^T \mathbf{y}_{\ell}(t) + ad\mathbf{u}(t) \approx \sum_{\ell=1}^2 a^{\ell} \mathbf{c}^T \mathbf{y}_{\ell}(t) + ad\mathbf{u}(t), \\ &= \int_0^t h_1(\tau_1) \mathbf{u}(t - \tau_1) d\tau_1 + \int_0^t \int_0^{t-\tau_2} h_2(\tau_1, \tau_2) \mathbf{u}(t - \tau_1 - \tau_2) \mathbf{u}(t - \tau_2) d\tau_1 d\tau_2 \\ &\quad + \int_0^t \int_0^{t-\tau_3} \int_0^{t-\tau_3} h_3(\tau_1, \tau_2, \tau_3) \mathbf{u}(t - \tau_1 - \tau_3) \mathbf{u}(t - \tau_2 - \tau_3) d\tau_1 d\tau_2 d\tau_3, \end{aligned}$$

with kernels

$$h_1(\tau_1) = \mathbf{c}^T e^{\mathbf{A}\tau_1} \mathbf{b},$$

$$h_2(\tau_1, \tau_2) = \mathbf{c}^T e^{\mathbf{A}\tau_2} \mathbf{N} e^{\mathbf{A}\tau_1} \mathbf{b},$$

$$h_3(\tau_1, \tau_2, \tau_3) = \mathbf{c}^T e^{\mathbf{A}\tau_3} \mathbf{G}(e^{\mathbf{A}\tau_2} \mathbf{b}, e^{\mathbf{A}\tau_1} \mathbf{b}).$$

Define

$$\Phi(s) = (s\mathbf{E} - \mathbf{A})^{-1}.$$

Levels of transfer functions:

$$\mathbf{H}_0(s) = \mathbf{c}^T \Phi(s) \mathbf{b},$$

$$\mathbf{H}_1(s_0, s_1) = \mathbf{c}^T \Phi(s_0) \mathbf{N} \Phi(s_1) \mathbf{b},$$

$$\mathbf{H}_2(s_0, s_1, s_2) = \mathbf{c}^T \Phi(s_0) \mathbf{G}(\Phi(s_1) \mathbf{b}, \Phi(s_2) \mathbf{b}),$$

⋮

# Loewner ROM

Frequency data:  $k = 3\bar{k}$

$$\mu_1^{(1)}, \mu_2^{(1)}, \mu_3^{(1)}, \dots, \mu_1^{(\bar{k})}, \mu_2^{(\bar{k})}, \mu_3^{(\bar{k})}, \quad \lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}, \dots, \lambda_1^{(\bar{k})}, \lambda_2^{(\bar{k})}, \lambda_3^{(\bar{k})}.$$

Grouped into multi-tuples

$$\begin{aligned}\boldsymbol{\mu}^{(j)} &= \{(\mu_1^{(j)}), (\mu_1^{(j)}, \mu_2^{(j)}), (\mu_1^{(j)}, \lambda_1^{(j)}, \mu_3^{(j)})\}, \quad j = 1, \dots, \bar{k}, \\ \boldsymbol{\lambda}^{(j)} &= \{(\lambda_1^{(j)}), (\lambda_2^{(j)}, \lambda_1^{(j)}), (\lambda_3^{(j)}, \lambda_1^{(j)}, \lambda_1^{(j)})\}, \quad j = 1, \dots, \bar{k}.\end{aligned}$$

Generalized controllability matrix  $\mathcal{R} \in \mathbb{C}^{n \times k}$

$$\mathcal{R} = [\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \dots, \mathcal{R}^{(\bar{k})}] \in \mathbb{C}^{n \times k},$$

$$\mathcal{R}^{(j)} = [\Phi(\lambda_1^{(j)}) \mathbf{b}, \Phi(\lambda_2^{(j)}) \mathbf{N} \Phi(\lambda_1^{(j)}) \mathbf{b}, \Phi(\lambda_3^{(j)}) \mathbf{G} (\Phi(\lambda_1^{(j)}) \mathbf{b}, \Phi(\lambda_1^{(j)}) \mathbf{b})].$$

Generalized observability matrix

$$\mathcal{O} = [(\mathcal{O}^{(1)})^T, (\mathcal{O}^{(2)})^T, \dots, (\mathcal{O}^{(\bar{k})})^T]^T \in \mathbb{C}^{k \times n},$$

$$\mathcal{O}^{(j)} = \begin{bmatrix} \mathbf{c}^T \Phi(\mu_1^{(j)}) \\ \mathbf{c}^T \Phi(\mu_1^{(j)}) \mathbf{N} \Phi(\mu_2^{(j)}) \\ \mathbf{c}^T \Phi(\mu_1^{(j)}) \mathbf{G} (\Phi(\lambda_1^{(j)}) \mathbf{b}, \Phi(\mu_3^{(j)})) \end{bmatrix}.$$

# Loewner ROM

- ▶ Loewner matrix  $\mathbb{L}$  and shifted Loewner matrix  $\mathbb{L}_s$

$$\mathbb{L} = -\mathcal{O} \mathbf{E} \mathcal{R}, \quad \mathbb{L}_s = -\mathcal{O} \mathbf{A} \mathcal{R}.$$

- ▶ Compute (short) SVDs

$$[\mathbb{L} \quad \mathbb{L}_s] = \mathbf{Y}_1 \Sigma_1 \mathbf{X}_1^*, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} = \mathbf{Y}_2 \Sigma_2 \mathbf{X}_2^*,$$

where  $\Sigma_1 \in \mathbb{R}^{k \times 2k}$ ,  $\Sigma_2 \in \mathbb{R}^{2k \times k}$ ,  $\mathbf{Y}_1, \mathbf{X}_2 \in \mathbb{C}^{k \times k}$ .

- ▶  $\mathbf{Y}, \mathbf{X} \in \mathbb{C}^{k \times r}$  obtained by selecting first  $r$  columns of  $\mathbf{Y}_1$  and  $\mathbf{X}_2$ .  
Loewner ROM is

$$\begin{aligned}\hat{\mathbf{E}} &= -\mathbf{Y}^* \mathbb{L} \mathbf{X} = \mathbf{W}^* \mathbf{E} \mathbf{V}, & \hat{\mathbf{A}} &= -\mathbf{Y}^* \mathbb{L}_s \mathbf{X}^* = \mathbf{W}^* \mathbf{E} \mathbf{V}, \\ \hat{\mathbf{G}}(\cdot, \cdot) &= \mathbf{W}^* \mathbf{G}(\mathbf{V} \cdot, \mathbf{V} \cdot), & \hat{\mathbf{N}} &= \mathbf{W}^* \mathbf{N} \mathbf{V}, \\ \hat{\mathbf{b}} &= \mathbf{W}^* \mathbf{b}, & \hat{\mathbf{c}} &= \mathbf{V}^* \mathbf{c}.\end{aligned}$$

- ▶ ROM  $\hat{\mathbf{E}}, \hat{\mathbf{A}}, \dots$ , represented by projection, but can compute ROM directly from (sub-)transfer function measurements (no need to explicitly project FOM).
- ▶ ROM interpolates (sub-)transfer functions.

- ▶ Convergence of expansion for  $a \in (0, a_{\max})$  proven for finite dimensional, so-called linear analytic state equations (quadratic bilinear systems are a special case) in, e.g., [Rugh, 1981].
- ▶ Proof applied to finite dimensional quadratic-bilinear systems would be useful (characterization of  $a_{\max}$ ), as well as extension to PDEs.
- ▶ Error estimate for truncation needed.  
Truncation at  $\ell = 3$  somewhat arbitrary, but seems to work well in practice.  
Approach can be extended to truncation at  $\ell > 3$ .
- ▶ Previous derivation assumes  $\mathbf{E}$  is invertible.  
Not true for Navier-Stokes.  
Use (discrete) Leray projections [Heinkenschloss et al., 2008] to extend technique (in progress).

## Burger's Equation

Viscosity  $\nu > 0$ , parameters  $\sigma_0 \leq 0$ ,  $\sigma_1 \geq 0$  ( $\nu = 0.01$ ,  $\sigma_0 = 0$ ,  $\sigma_1 = 0.1$ ),

$$\begin{aligned} \frac{\partial}{\partial t}y(x,t) - \nu \frac{\partial^2}{\partial x^2}y(x,t) + y(x,t) \frac{\partial}{\partial x}y(x,t) &= 0, & x \in (0,1), t \in (0,T), \\ \nu \frac{\partial}{\partial x}y(x,0) + \sigma_0 y(0,t) &= u(t), & t \in (0,T), \\ \nu \frac{\partial}{\partial x}y(x,1) + \sigma_1 y(1,t) &= 0, & t \in (0,T), \\ y(x,0) &= 0, & x \in (0,1). \end{aligned}$$

Input  $u$ ; output

$$z(t) = \int_0^1 y(x,t) dx.$$

Finite element semi-discretization leads to

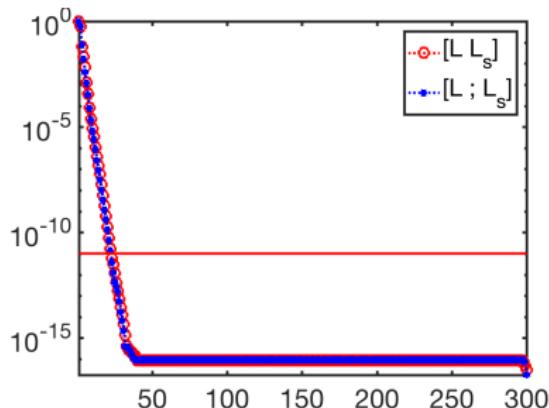
$$\begin{aligned} \mathbf{E} \frac{d}{dt} \mathbf{y}(t) &= \mathbf{A} \mathbf{y}(t) + \mathbf{G}(\mathbf{y}(t), \mathbf{y}(t)) + \mathbf{N} \mathbf{y}(t) \mathbf{u}(t) + \mathbf{b} \mathbf{u}(t), & t \in (0,T), \\ \mathbf{z}(t) &= \mathbf{c}^T \mathbf{y}(t), & t \in (0,T), \\ \mathbf{y}(0) &= \mathbf{0}. \end{aligned}$$

Computations with piecewise linear FEM, FOM size  $n = 257$ .

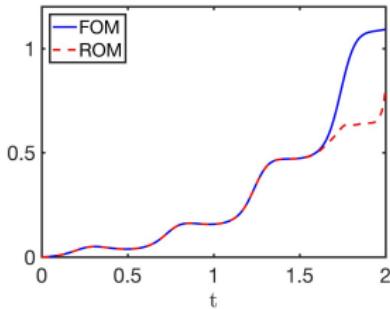
Use the input  $u(t) = 0.1 \sin(4\pi t)$  in simulations.

Frequency data: 300 logarithmically spaced points between 1 and  $10^3$  and their complex conjugates (will be able to construct real ROMs).

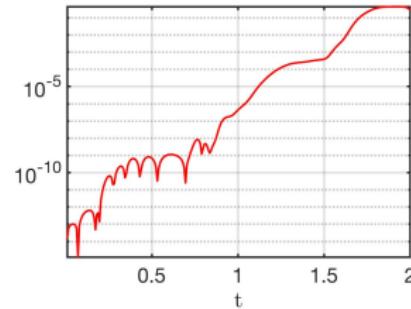
Split into  $\mu_1, \dots, \mu_k, \lambda_1, \dots, \lambda_k \in \mathbb{C}$ ,  $k = 300$ .



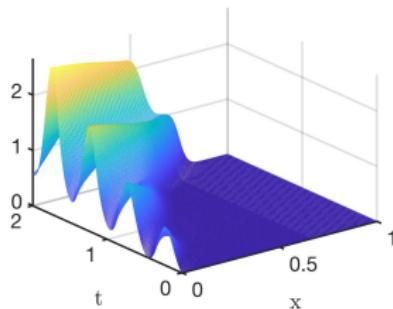
Normalized singular values of the Loewner matrices  $[\mathbb{L} \quad \mathbb{L}_s]$  and  $\begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix}$ .  
ROM size  $r = 22$  is chosen to be smallest  $r$  with  $\sigma_r/\sigma_1 > 10^{-11}$ .



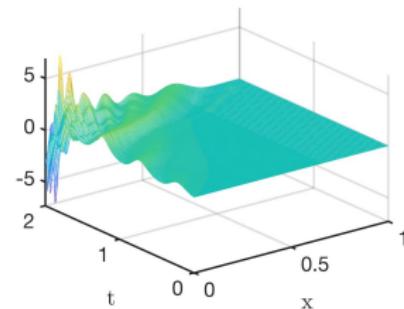
Output of FOM and Loewner ROM



Error between FOM and Loewner ROM outputs



State computed with FOM



State computed with Loewner ROM

Instabilities may arise because Loewner ROM corresponds to Petrov-Galerkin Projection  $\mathbf{V} \neq \mathbf{W}$ .

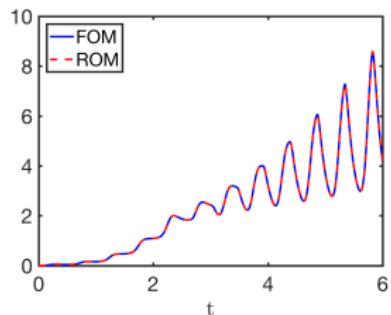
- ▶ Can show existence of constant  $C > 0$  that depends on  $T, \nu > 0$ , but not on  $u$  such that

$$\|y\|_{W(0,T)} + \|y\|_{L^\infty} \leq C(1 + \|u\|_{L^2(0,T)}). \quad (*)$$

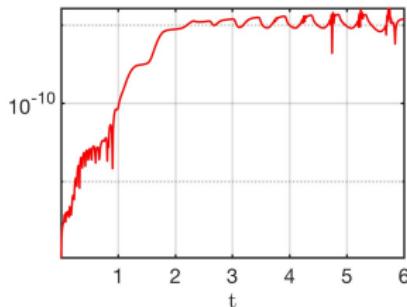
- ▶ Similar estimate holds for any **Galerkin** approximation.
- ▶ However, Loewner ROM is a **Petrov-Galerkin** approximation. Can lead to instabilities - as on previous slide.
- ▶ Use Loewner ROM projection matrices  $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ , construct  $[\mathbf{V}, \mathbf{W}] \in \mathbb{R}^{n \times 2r}$  (orthogonalize), and use this matrix to construct Galerkin projection ROM.

This ROM solution obeys estimate like (\*).

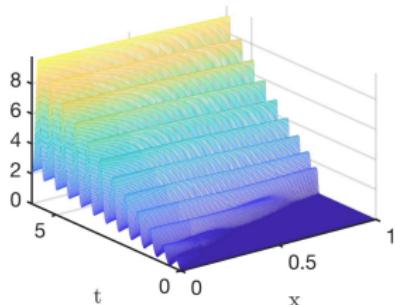
# Use Projection Matrix $[V, W]$



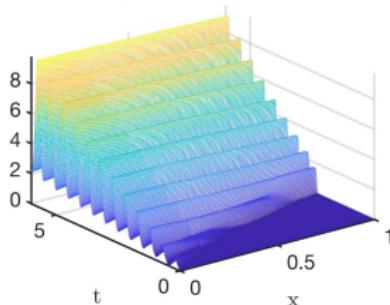
Output of FOM and Loewner ROM



Error between FOM and Loewner ROM outputs



State computed with FOM



State computed with Loewner ROM

Excellent agreement between FOM and Loewner ROM  
(with projection matrix  $[V, W]$ )

## Conclusions:

- ▶ Loewner framework to construct ROM directly from data.
  - ▶ Theoretically, Loewner is a projection based, interpolatory ROM
  - ▶ Can do so for LTI systems.
  - ▶ Can be extended to quadratic-bilinear systems, but not clear how data can be directly obtained.
- ▶ Loewner can be extended to quadratic-bilinear systems, but not clear how data can be directly obtained.
- ▶ Error estimates for quadratic-bilinear systems.
- ▶ Stability of Loewner for quadratic-bilinear systems.

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